Mock Exam algebraic topology October 2024

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

1. Explicitly write down two ASC Σ and Φ such that $\Sigma \approx \Phi$ but Σ is not simplicially isomorphic to Φ.

2. Does there exist a connected surface Σ without boundary with Euler characteristic -3? (By a surface we mean a two-dimensional ASC such that the star of every vertex is simplicially isomorphic to a cone on a circle). If yes sketch how Σ can be constructed, if no explain why it is not possible.

3. Consider the ASC $\Sigma = \langle \{\{k, k+1\} | k \in \mathbb{Z}/3n\mathbb{Z}\} \rangle$ and the map $f : \Sigma \to \mathbb{S}^1$ defined by $f(3s + j) = j$. Describe the image of $f_* : H_1(\Sigma) \to H_1(\mathbb{S}^1)$.

4. Suppose we have two disjoint ASC Φ and Σ such that $\sigma \in \Sigma^{(20)}$ and $\tau \in \Phi^{(24)}$ and consider the ASC $\Gamma = \Phi \cup \Sigma \cup (\langle \sigma \rangle \times \langle \tau \rangle)$. For which $n \in \mathbb{N}$ do we have $H_{2024}(\Gamma) \cong H_{2024}(\Sigma) \times H_{2024}(\Phi) \times \mathbb{Z}^n$?

5. Suppose $f : \Gamma \to \Sigma$ is a regular covering map with automorphism group A and B is a normal subgroup of A. Show that there exists a regular covering $g : \Phi \to \Sigma$ with automorphism group A/B.

6. Denote the closed line segment in $\mathbb{C}$ connecting the points x, y by $L(x, y)$. Prove that the following subsets of $\mathbb{C}$ taken with the subspace topology are homotopy equivalent: $A = L(-i, i) \cup L(i, 2 - i) \cup L(0, 1)$ and $D = L(-i, i) \cup L(i, 1) \cup L(-i, 1)$. You are not required to explicitly prove that all the maps you are constructing are continuous.

7. Suppose Σ is a finite ASC with $b \in V(\Sigma)$ and $\dim H_1(\Sigma; \mathbb{Q}) = 6$ and X is a finite set on which group G acts from the right and $z : C_1(\Sigma) \to G$ is a constructor function. Show that $[u, v].\alpha = [v\alpha, w\alpha]$, where $\alpha \in \pi_1(\Sigma, b)$ defines an action of $\pi_1(\Sigma, b)$ on $H_1(\Sigma_z(X); \mathbb{Q})$.

8. Suppose we have two disjoint connected and orientable surfaces A and B inside the n-th barycentric subdivision of $\mathbb{S}^3$. If the boundaries of A and B form a two component link then show this link must have linking number zero.

9. Show that there exist infinitely many pair-wise non-homotopic maps $|\mathbb{S}^1| \to |\mathbb{S}^1|$ and that there are at most 27 simplicial maps $\mathbb{S}^1 \to \mathbb{S}^1$. Why does this not contradict the simplicial approximation theorem?